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Ordered sets and operations "negation" over their elements

Krassimir Atanassov

Department of Bioinformatics and Mathematical Modelling,
Institute of Biophysics and Biomedical Engineering,
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105, Sofia-1113, Bulgaria,
e-mail: *krat@bas.bg*
and
Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Bourgas-8010, Bulgaria

Abstract. The idea for defining of operations "negation" over ordered sets is discussed. Five types of negations are introduced and their basic properties are studied.

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1 Introduction

Some months ago, when I started work on this theme, I had in mind temporal scales, that can be used in Temporal Intuitionistic Fuzzy Sets (TIFSs; see [1, 2]). By this reason, in all places the the elements of the fixed ordered set are called "time-moments". Later, I saw that the research has more global character and I changed the words "temporal scale(s)" with "(well-)orderd set(s)". Really, the results of the present research will be used in a future research on TIFSs, but now, it is clear that they can obtain other applications, too.

Bellow, we will discuss five different operations "negation" over ordered sets.

2 Main results

Let $T = [T_0, T_1]$ be a well-ordered set, where $T_0 < T_1$. Let us define

$$T_\tau = \{t | t \in T \ \&\ t \geq \tau\},$$

$$T^\tau = \{t | t \in T \ \&\ t \leq \tau\}.$$

Therefore,

$$T_\tau \cup T^\tau = T,$$

$$T_\tau \cap T^\tau = \{\tau\},$$

where $\cap$ and $\cup$ are the set-theoretical operations "union" and "intersection".

Let $t \in T$ be a fixed time-moment. The geometrical interpretation of the scale is given on Fig. 1.

Fig. 1.

2.1 Boundary-symmetric negation

For each $t \in T$, we define

$$\sim_1 t = T_0 + T_1 - t.$$

Then, for each $t \in T : T_0 \leq \sim_1 t \leq T_1$ and

$$\sim_1 T_0 = T_0 + T_1 - T_0 = T_1,$$

$$\sim_1 T_1 = T_0 + T_1 - T_1 = T_0,$$

$$\sim_1 \frac{T_0 + T_1}{2} = T_0 + T_1 - \frac{T_0 + T_1}{2} = \frac{T_0 + T_1}{2},$$

$$\sim_1 \sim_1 t = \sim_1 (T_0 + T_1 - \sim_1 t) = T_0 + T_1 - (T_0 + T_1 - t) = t.$$

Therefore, operation boundary-symmetric negation is a negation from classical type, because it satisfies Aristotle's Law for Excluded Middle (see, e.g., [4]).

The geometrical interpretation of the boundary-symmetric negation is given on Fig. 2.

Fig. 2.

2.2 Symmetric negation

Let $\tau \in T$ be a fixed and let $T_0 < \tau < T_1$.

For each $t \in T$, we define

$$\sim_2 t = \begin{cases} \min(T_1, 2\tau - t), & \text{if } t \in [T_0, \tau] \\ \max(T_0, 2\tau - t), & \text{if } t \in [\tau, T_1] \end{cases}$$

Then, for each $t \in T : T_0 \leq \sim_1 t \leq T_1$ and

$$\sim_2 T_0 = \min(T_1, 2\tau - T_0) \leq T_1,$$

$$\sim_2 T_1 = \max(T_0, 2\tau - T_0) \geq T_0,$$

$$\sim_2 \tau = \min(T_1, 2\tau - \tau) = \min(T_1, \tau) = \tau,$$

$$\sim_2 \tau = \max(T_0, 2\tau - \tau) = \max(T_0, \tau) = \tau.$$

Let $t \in [T_0, \tau]$. Then,

$$\sim_2 \sim_2 t = \sim_2 \min(T_1, 2\tau - t)$$

(because $\min(T_1, 2\tau - t) \in [\tau, T_1]$)

$$= \max(T_0, 2\tau - \min(T_1, 2\tau - t)) = \max(T_0, \max(2\tau - T_1, 2\tau - (2\tau - t)))$$

$$= \max(T_0, \max(2\tau - T_1, t)) = \max(T_0, 2\tau - T_1, t)$$

(because $t \geq T_0$)

$$= \max(2\tau - T_1, t) \geq t.$$

Let $t \in [\tau, T_1]$. Then,

$$\sim_2\sim_2 t = \sim_2 \max(T_0, 2\tau - t)$$

(because $\max(T_0, 2\tau - t) \in [T_0, \tau]$)

$$= \min(T_1, 2\tau - \max(T_0, 2\tau - t)) = \min(T_1, \min(2\tau - T_0, 2\tau - 2\tau + t))$$

$$= \min(T_1, \min(2\tau - T_0, t)) = \min(T_1, 2\tau - T_0, t)$$

(because $t \leq T_1$)

$$= \min(2\tau - T_0, t) \leq t.$$

The geometrical interpretation of the Symmetric negation is given on Fig. 3.

Fig. 3.

The operation Symmetric negation is a negation from a non-classical type, because it does not satisfy Aristotle's Law for Excluded Middle.

2.3 Proportional negation

Let $\tau \in T$ be a fixed and let $T_0 < \tau < T_1$.

For each $t \in T$, we define

$$\sim_3 t = \begin{cases} \tau + \frac{T_1-\tau}{\tau-T_0}.(\tau - t), & \text{if } t \in [T_0, \tau] \\ \\ \tau - \frac{\tau-T_0}{T_1-\tau}.(t - \tau), & \text{if } t \in [\tau, T_1] \end{cases}$$

First, we check that for $t \in [T_0, \tau]$:

$$T_1 - \tau - \frac{T_1 - \tau}{\tau - T_0}.(\tau - t) = \frac{T_1 - \tau}{\tau - T_0}.(\tau - T_0 - \tau + t)$$

$$= \frac{T_1 - \tau}{\tau - T_0}.(t - T_0) > 0$$

and for $t \in [\tau, T_1]$:

$$\tau - \frac{\tau - T_0}{T_1 - \tau}.(t - \tau) - T_0 = \frac{\tau - T_0}{T_1 - \tau}(T_1 - \tau - t + \tau)$$

$$= \frac{\tau - T_0}{T_1 - \tau}(T_1 - t) > 0,$$

i.e., the definition is correct.

Then,

$$\sim_3 T_0 = \tau + \frac{T_1 - \tau}{\tau - T_0}.(\tau - T_0) = \tau + T_1 - \tau = T_1,$$

$$\sim_3 T_1 = \tau - \frac{\tau - T_0}{T_1 - \tau}.(T_1 - \tau) = \tau - T_0 + \tau = T_0,$$

$$\sim_3 \tau = \tau + \frac{T_1 - \tau}{\tau - T_0}.(\tau - \tau) = \tau,$$

$$\sim_3 \tau = \tau - \frac{\tau - T_0}{T_1 - \tau}.(\tau - \tau) = \tau.$$

Let $t \in [T_0, \tau]$. Then,

$$\sim_3\sim_3 t = \sim_3 \left(\tau + \frac{T_1 - \tau}{\tau - T_0}.(\tau - t)\right)$$

(because $\tau + \frac{T_1-\tau}{\tau-T_0}.(\tau - t) \in [\tau, T_1]$)

$$= \tau - \frac{\tau - T_0}{T_1 - \tau}.\left(\tau + \frac{T_1 - \tau}{\tau - T_0}.(\tau - t) - \tau\right)$$

$$= \tau - \frac{\tau - T_0}{T_1 - \tau}.\frac{T_1 - \tau}{\tau - T_0}.(\tau - t) = t.$$

Let $t \in [\tau, T_1]$. Then,

$$\sim_3\sim_3 t = \sim_3 \left(\tau - \frac{\tau - T_0}{T_1 - \tau}.(t - \tau)\right)$$

(because $\tau + \frac{T_1-\tau}{\tau-T_0}.(\tau - t) \in [T_0, \tau]$)

$$= \tau + \frac{T_1 - \tau}{\tau - T_0}.\left(\tau - \left(\tau - \frac{\tau - T_0}{T_1 - \tau}.(t - \tau)\right)\right)$$

$$= \tau + \frac{T_1 - \tau}{\tau - T_0}.\frac{\tau - T_0}{T_1 - \tau}.(t - \tau) = t.$$

The geometrical interpretation of the Proportional negation is given on Fig. 4.

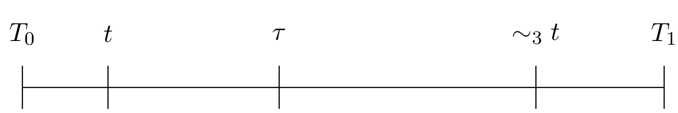

Fig. 4.

The operation Proportional negation is a negation from a classical type.

Let

$$\mathrm{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}.$$

Then, we can combine the two forms of this operation to the form:

$$\sim_3 t = \tau + \left(\frac{T_1 - \tau}{\tau - T_0}\right)^{\mathrm{sign}(\tau - t)} .(\tau - t).$$

Really, if $t < \tau$ then $\mathrm{sign}(\tau - t) = 1$ and

$$\sim_3 t = \tau + \left(\frac{T_1 - \tau}{\tau - T_0}\right) .(\tau - t),$$

if $t = \tau$ then $\mathrm{sign}(\tau - t) = 0$ and

$$\sim_3 t = \tau + \tau - t = t,$$

while, if $t > \tau$ then $\mathrm{sign}(\tau - t) = -1$ and

$$\begin{aligned} \sim_3 t &= \tau + \left(\frac{T_1 - \tau}{\tau - T_0}\right)^{-1} .(\tau - t) \\ &= \tau + \left(\frac{\tau - T_0}{T_1 - \tau}\right) .(\tau - t) \\ &= \tau - \left(\frac{\tau - T_0}{T_1 - \tau}\right) .(t - \tau). \end{aligned}$$

2.4 First proportional-interval negation

Let $\tau', \tau'' \in T$ be a fixed and let $T_0 < \tau' < \tau'' < T_1$.

For each $t \in T$, we define

$$\sim_4 t = \begin{cases} \tau'' + \frac{T_1 - \tau''}{\tau' - T_0}.(\tau' - t), & \text{if } t \in [T_0, \tau'] \\ \\ \tau' - \frac{\tau' - T_0}{T_1 - \tau''}.(t - \tau''), & \text{if } t \in [\tau'', T_1] \end{cases}.$$

The check for the correctness of the definition is similar to the check in Subsection 2.3. Really, for $t \in [T_0, \tau]$:

$$T_1 - \tau'' - \frac{T_1 - \tau''}{\tau' - T_0}.(\tau' - t) = \frac{T_1 - \tau''}{\tau' - T_0}.(t - T_0) > 0$$

and for $t \in [\tau, T_1]$:

$$\tau' - \frac{\tau' - T_0}{T_1 - \tau''}.(t - \tau'') - T_0 = \frac{\tau' - T_0}{T_1 - \tau''}(T_1 - t) > 0,$$

i.e., the definition is correct.

Then,

$$\sim_4 T_0 = \tau'' + \frac{T_1 - \tau''}{\tau' - T_0}.(\tau' - T_0) = T_1,$$

$$\sim_4 T_1 = \tau' - \frac{\tau' - T_0}{T_1 - \tau''}.(T_1 - \tau'') = T_0,$$

$$\sim_4 \tau' = \tau'' + \frac{T_1 - \tau''}{\tau' - T_0}.(\tau' - \tau') = \tau'',$$

$$\sim_4 \tau'' = \tau' - \frac{\tau' - T_0}{T_1 - \tau''}.(t'' - \tau'') = \tau'.$$

Let $t \in [T_0, \tau']$. Then,

$$\sim_4 \sim_4 t = \sim_4 \left(\tau'' + \frac{T_1 - \tau''}{\tau' - T_0}.(\tau' - t)\right)$$

(because $\tau'' + \frac{T_1 - \tau''}{\tau' - T_0}.(\tau' - t) \in [\tau'', T_1]$)

$$= \tau' - \frac{\tau' - T_0}{T_1 - \tau''}.\left(\left(\tau'' + \frac{T_1 - \tau''}{\tau' - T_0}.(\tau' - t)\right) - \tau''\right)$$

$$= \tau' - \frac{\tau' - T_0}{T_1 - \tau''}.\frac{T_1 - \tau''}{\tau' - T_0}.(\tau' - t) = t.$$

Let $t \in [\tau'', T_1]$. Then,

$$\sim_4 \sim_4 t = \sim_4 \left(\tau' - \frac{\tau' - T_0}{T_1 - \tau''}.(t - \tau'')\right)$$

(because $\tau' - \frac{\tau' - T_0}{T_1 - \tau''}.(t - \tau'') \in [T_0, \tau']$)

$$= \tau'' + \frac{T_1 - \tau''}{\tau' - T_0}.\left(\tau' - \left(\tau' - \frac{\tau' - T_0}{T_1 - \tau''}.(t - \tau'')\right)\right)$$

$$= \tau'' + \frac{T_1 - \tau''}{\tau' - T_0}.\frac{\tau' - T_0}{T_1 - \tau''}.(t - \tau'') = t.$$

The geometrical interpretation of the first proportional-interval negation is given on Fig. 5.

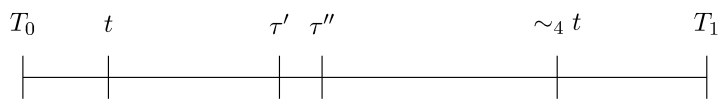

Fig. 5.

The operation first proportional-interval negation is a negation from a classical type.

2.5 Second proportional-interval negation

Let $\tau', \tau'' \in T$ be a fixed and let $T_0 < \tau' < \tau'' < T_1$.

For each $t \in T$, we define

$$\sim_5 t = \begin{cases} \tau'' + \frac{T_1-\tau''}{\tau''-T_0}.(\tau''-t), & \text{if } t \in [T_0, \tau'] \\ \\ \tau' - \frac{\tau'-T_0}{T_1-\tau'}.(t-\tau'), & \text{if } t \in [\tau'', T_1] \end{cases}.$$

The check for the correctness of the definition is similar to the checks in Subsections 2.3 and 2.4.

Then,

$$\sim_5 T_0 = \tau'' + \frac{T_1-\tau''}{\tau''-T_0}.(\tau''-T_0) = T_1,$$

$$\sim_5 T_1 = \tau' - \frac{\tau'-T_0}{T_1-\tau'}.(T_1-\tau') = T_0,$$

$$\sim_5 \tau' = \tau'' + \frac{T_1-\tau''}{\tau''-T_0}.(\tau''-\tau')$$

$$= \frac{1}{\tau''-T_0}.((\tau'')^2 - T_0\tau'' + T_1\tau'' - (\tau'')^2 - T_1\tau' + \tau'\tau'')$$

$$= \frac{1}{\tau''-T_0}.((T_1-T_0)\tau'' - (T_1-\tau'')\tau')$$

$$> \frac{\tau''}{\tau''-T_0}.(T_1-T_0-T_1+\tau'')$$

$$= \frac{\tau''}{\tau''-T_0}.(\tau''-T_0) = \tau'',$$

i.e., $\sim_5 \tau' > \tau''$.

$$\sim_5 \tau'' = \tau' - \frac{\tau'-T_0}{T_1-\tau'}.(\tau''-\tau')$$

$$= \frac{1}{T_1-\tau'}.(T_1\tau' - (\tau')^2 - \tau'\tau'' + T_0\tau'' + (\tau')^2 - T_0\tau')$$

$$= \frac{1}{T_1-\tau'}.((T_1-T_0)\tau' - (\tau'-T_0)\tau'')$$

$$< \frac{\tau'}{T_1-\tau'}.(T_1-T_0-\tau'+T_0)$$

$$= \frac{\tau'}{T_1-\tau'}.(T_1-\tau') = \tau',$$

i.e., $\sim_5 \tau'' < \tau'$.

Let $t \in [T_0, \tau']$. Then,

$$\sim_5\sim_5 t = \sim_5 \left(\tau'' + \frac{T_1-\tau''}{\tau''-T_0}.(\tau''-t)\right)$$

(because $\tau'' + \frac{T_1-\tau''}{\tau''-T_0}.(\tau''-t) \in [\tau'', T_1]$)

$$= \tau' - \frac{\tau'-T_0}{T_1-\tau'}.\left(\tau'' + \frac{T_1-\tau''}{\tau''-T_0}.(\tau''-t) - \tau'\right)$$

$$= \frac{\tau'-T_0}{T_1-\tau'}.\frac{T_1-\tau''}{\tau''-T_0}.t + \tau' - \frac{\tau'-T_0}{T_1-\tau'}.\left(\tau'' + \frac{T_1-\tau''}{\tau''-T_0}.\tau'' - \tau'\right)$$

$$= \frac{\tau'-T_0}{T_1-\tau'}.\frac{T_1-\tau''}{\tau''-T_0}.t + \left(1 + \frac{\tau'-T_0}{T_1-\tau'}\right)\tau' - \frac{\tau'-T_0}{T_1-\tau'}.\left(1 + \frac{T_1-\tau''}{\tau''-T_0}\right).\tau''$$

$$= \frac{\tau'-T_0}{T_1-\tau'}.\frac{T_1-\tau''}{\tau''-T_0}.t + \frac{T_1-T_0}{T_1-\tau'}.\tau' - \frac{\tau'-T_0}{T_1-\tau'}.\frac{T_1-T_0}{\tau''-T_0}.\tau''$$

$$= \frac{\tau'-T_0}{T_1-\tau'}.\frac{T_1-\tau''}{\tau''-T_0}.t + \frac{T_1-T_0}{T_1-\tau'}.\left(\frac{\tau'-T_0}{\tau''-T_0}.\tau'' - \tau'\right)$$

$$= \frac{\tau'-T_0}{T_1-\tau'}.\frac{T_1-\tau''}{\tau''-T_0}.t + \frac{T_1-T_0}{T_1-\tau'}.\frac{\tau'-\tau''}{\tau''-T_0}.T_0.$$

Now, we check that

$$t - \sim_5\sim_5 t = t - \frac{\tau'-T_0}{T_1-\tau'}.\frac{T_1-\tau''}{\tau''-T_0}.t - \frac{T_1-T_0}{T_1-\tau'}.\frac{\tau'-\tau''}{\tau''-T_0}.T_0$$

$$= \left(1 - \frac{\tau'-T_0}{T_1-\tau'}.\frac{T_1-\tau''}{\tau''-T_0}\right).t - \frac{T_1-T_0}{T_1-\tau'}.\frac{\tau'-\tau''}{\tau''-T_0}.T_0$$

$$= \frac{\tau''-\tau'}{T_1-\tau'}.\frac{T_1-T_0}{\tau''-T_0}.t - \frac{T_1-T_0}{T_1-\tau'}.\frac{\tau'-\tau''}{\tau''-T_0}.T_0$$

$$= \frac{\tau''-\tau'}{T_1-\tau'}.\frac{T_1-T_0}{\tau''-T_0}.(t-T_0) \geq 0.$$

Let $t \in [\tau'', T_1]$. Then,

$$\sim_5\sim_5 t = \sim_5 \left(\tau' - \frac{\tau'-T_0}{T_1-\tau'}.(t-\tau')\right)$$

(because $\tau' - \frac{\tau'-T_0}{T_1-\tau'}.(t-\tau') \in [T_0, \tau']$)

$$= \tau'' + \frac{T_1-\tau''}{\tau''-T_0}.\left(\tau'' - \left(\tau' - \frac{\tau'-T_0}{T_1-\tau'}.(t-\tau')\right)\right)$$

$$= \left(1 + \frac{T_1-\tau''}{\tau''-T_0}\right).\tau'' - \frac{T_1-\tau''}{\tau''-T_0}.\left(1 + \frac{\tau'-T_0}{T_1-\tau'}\right).\tau' + \frac{T_1-\tau''}{\tau''-T_0}.\frac{\tau'-T_0}{T_1-\tau'}.t$$

$$= \frac{T_1-\tau''}{\tau''-T_0}.\frac{\tau'-T_0}{T_1-\tau'}.t + \frac{T_1-T_0}{\tau''-T_0}.\tau'' - \frac{T_1-\tau''}{\tau''-T_0}.\frac{T_1-T_0}{T_1-\tau'}.\tau'$$

$$= \frac{T_1-\tau''}{\tau''-T_0}.\frac{\tau'-T_0}{T_1-\tau'}.t + \frac{T_1-T_0}{\tau''-T_0}.\left(\tau'' - \frac{T_1-\tau''}{T_1-\tau'}.\tau'\right)$$

$$= \frac{T_1-\tau''}{\tau''-T_0}.\frac{\tau'-T_0}{T_1-\tau'}.t + \frac{T_1-T_0}{\tau''-T_0}.\frac{T_1-T_0}{T_1-\tau'}.T_1.$$

Now, we check that

$$\begin{aligned}
\sim_5\sim_5 t - t &= \frac{T_1-\tau''}{\tau''-T_0}.\frac{\tau'-T_0}{T_1-\tau'}.t + \frac{T_1-T_0}{\tau''-T_0}.\frac{T_1-T_0}{T_1-\tau'}.T_1 - t \\
&= \left(\frac{T_1-\tau''}{\tau''-T_0}.\frac{\tau'-T_0}{T_1-\tau'} - 1\right).t + \frac{T_1-T_0}{\tau''-T_0}.\frac{T_1-T_0}{T_1-\tau'}.T_1 \\
&= -\frac{T_1-T_0}{\tau''-T_0}.\frac{T_1-T_0}{T_1-\tau'}.t + \frac{T_1-T_0}{\tau''-T_0}.\frac{T_1-T_0}{T_1-\tau'}.T_1 \\
&= \frac{T_1-T_0}{\tau''-T_0}.\frac{T_1-T_0}{T_1-\tau'}(T_1-t) \geq 0.
\end{aligned}$$

Finally, we see that for each $t \in [T_0, \tau']$:

$$\begin{aligned}
\sim_5 t - \sim_4 t &= \tau'' + \frac{T_1-\tau''}{\tau''-T_0}.(\tau''-t) - \tau'' - \frac{T_1-\tau''}{\tau'-T_0}.(\tau'-t) \\
&= \frac{T_1-\tau''}{\tau''-T_0}.(\tau''-t) - \frac{T_1-\tau''}{\tau'-T_0}.(\tau'-t) \\
&= \frac{T_1-\tau''}{(\tau''-T_0)(\tau'-T_0)}.((\tau'-T_0)(\tau''-t) - (\tau''-T_0)(\tau'-t)) \\
&= \frac{T_1-\tau''}{(\tau''-T_0)(\tau'-T_0)}.(\tau''-\tau')(t-T_0) > 0
\end{aligned}$$

and for each $t \in [\tau'', T_1]$:

$$\begin{aligned}
\sim_4 t - \sim_5 t &= \tau' - \frac{\tau'-T_0}{T_1-\tau''}.(t-\tau'') - \tau' + \frac{\tau'-T_0}{T_1-\tau'}.(t-\tau') \\
&= \frac{\tau'-T_0}{T_1-\tau'}.(t-\tau') - \frac{\tau'-T_0}{T_1-\tau''}.(t-\tau'') \\
&= \frac{\tau'-T_0}{(T_1-\tau')(T_1-\tau'')}.((T_1-\tau'')(t-\tau') - (T_1-\tau')(t-\tau'')) \\
&= \frac{\tau'-T_0}{(T_1-\tau')(T_1-\tau'')}.(T_1-t)(\tau''-\tau') > 0.
\end{aligned}$$

The geometrical interpretation of the second proportional-interval negation is given on Fig. 6.

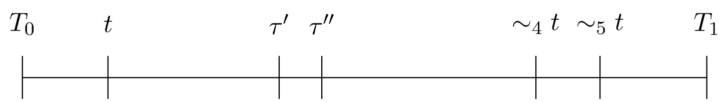

Fig. 6.

The operation second proportional-interval negation is a negation from a non-classical type.

3 Conclusion

As it was mentioned above, in near future, the results from the present paper will be used for definitions of new operators over TIFSs. The first step was done with [3], where the first temporal-modal operators were defined for the case of TIFSs.

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